On relativistic motion of a pair of particles having opposite signs of masses

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In this methodological note we consider, in a weak-field limit, a relativistic linear motion of two particles with opposite signs of masses and a small difference between their absolute values $m_{1,2} = \pm(\mu \pm \Delta\mu)$, $\mu > 0$, $|\Delta\mu| \ll \mu$. In 1957 H. Bondi showed both in framework of Newtonian analysis and in General Relativity that when the relative motion of particles is absent such a pair can be accelerated indefinitely. We generalise results of his paper to account for a small nonzero difference between velocities of the particles.

Assuming that the weak-field limit holds and the dynamical system is conservative an elementary treatment of the problem based on the laws of energy and momentum conservation shows that the system can be accelerated indefinitely, or attain very large asymptotic values of the Lorentz factor γ. The system experiences indefinite acceleration when its energy-momentum vector is null and the mass difference $\Delta\mu \le 0$. When modulus of the square of the norm of the energy-momentum vector, $|N^2|$, is sufficiently small the system can be accelerated to very large $\gamma \propto |N^2|^{-1}$.

It is stressed that when only leading terms in the ratio of a characteristic gravitational radius to the distance between the particles are retained our elementary analysis leads to equations of motion equivalent to those derived from relativistic weak-field equations of motion of Havas and Goldberg 1962.

Thus, in the weak-field approximation, it is possible to bring the system to the state with extremely high values of γ. The positive energy carried by the particle with positive mass may be conveyed to other physical bodies say, by intercepting this particle with a target. Suppose that there is a process of production of such pairs and the particles with positive mass are intercepted while the negative mass particles are expelled from the region of space occupied by physical bodies of interest. This scheme could provide a persistent transfer of positive energy to the bodies, which may be classified as a 'Perpetuum Motion of Third Kind'.

Additionally, we critically evaluate some recent claims on the problem.

I. INTRODUCTION

Bondi 1957 [1] pointed out that in the Newtonian approximation two particles with opposite signs of masses at rest with respect to each other accelerate indefinitely in an inertial frame. This process is allowed by the laws of conservation since the kinetic energy and angular momentum of such a system are conserved being exactly zero while the potential energy depends only on relative distance between the particles. In the same paper he generalised this result by finding an appropriate static accelerated solution in General Relativity and discovered that a uniformly accelerated pair of particles with opposite signs of masses must have a mass difference determined by the fact that constant in time particle accelerations must be different to keep them static with respect to each other.

It is trivial to show that in the Newtonian approximation (see the next Section) when the two particles with opposite signs of masses have a relative velocity, its value is approximately conserved. As a result, the acceleration period is finite and the pair as a whole being initially at rest gains a finite value of velocity. We also show that when initial relative velocity of the particles is sufficiently small the pair can be accelerated to a relativistic speed.

In the next Section we consider the problem in the relativistic setting and generalise Bondi's analysis considering pairs of particles with opposite masses and small difference between their absolute values: $m_{1,2} = \pm(\mu \pm \Delta\mu)$, $\mu > 0$, $|\Delta\mu| \ll \mu$ having an initial relative velocity v_{in} in a fixed lab frame where the pair as a whole is initially at rest. We assume that gravitational interaction is weak and, therefore, $G\mu/(c^2 D_{in}) \ll 1$, where D_{in} is an initial separation distance between the particles. Also, for simplicity, in the relativistic treatment, it is assumed that the orbital angular momentum of the system is equal to zero and the motion is linear.

We analyse this situation by elementary means. The equations of motion are obtained from the laws of energy and momentum conservation. It is assumed that energy and momentum of the system in a Lorentz frame instantaneously comoving with the motion of the pair are given by the Newtonian expressions and that they form time and spacial components of a local four-vector. Then this energy-momentum vector is projected onto the lab frame. Since energy

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and momentum in the lab frame are conserved under the assumption that gravitational radiation from the system is insignificant we get two equations of first order in time fully describing dynamics of the system. We also show how to derive an equivalent pair of second order equations considering the Newton's law of gravity in a frame accelerating with the particles.

It is shown that the pair as a whole always have a positive acceleration with its asymptotic value being either zero or a nonzero constant depending on initial conditions. The relative distance between the particles can either have a turning point or increase monotonically. The system accelerates indefinitely when the mass difference $\Delta\mu \leq 0$ and the norm of the energy-momentum vector $N = \sqrt{(2\Delta\mu c^2 + \frac{G\mu^2}{D_{in}})^2 - \mu^2 v_{in}^2} = 0$, and, accordingly, the energy-momentum vector is null. In this case the relative distance increases monotonically. When N^2 is sufficiently small, for the initial conditions corresponding to the monotonic behaviour of the relative distance the acceleration period is finite but the asymptotic value of the Lorentz gamma factor is large being proportional to $|N^2|^{-1}$.

Such pairs can play a role in realisation of a hypothetical effect, which we called 'Perpetuum Motion of Third Kind' [2], hereafter PMT. In its most general formulation this effect is a possibility of a persistent energy transfer from a subsystem having negative energy to a subsystem with positive energy, in classical theories where negative energy subsystems are possible. Indeed, the positive mass particle can, in principle, be used to transfer positive energy to other physical bodies after the pair has been accelerated to high values of the Lorentz factor. Iterating this process as many times as we need we can extract as much positive energy as we wish. Note, however, that it is not the only 'working model' of PMT, and that, in principal, in order to make PMT we need neither systems with negative rest mass nor gravitational interactions. As is shown in [2] it suffices to have a medium violating the weak energy condition with certain additional properties and mere hydrodynamical interaction 'to construct a PMT'.

Additionally, we comment on several statements of paper [3], where the Kepler problem for a binary with opposite signs of masses has been considered which may, in our opinion, lead to misunderstanding of the problem.

II. NEWTONIAN TREATMENT OF THE PROBLEM

At first let us consider the problem in the Newtonian approximation where mutual gravitational accelerations acting on the particles of masses m_1 and m_2 are given by the conventional expressions:

$$\ddot{\mathbf{r}}_1 = -\frac{Gm_2}{|\mathbf{D}|^3}\mathbf{D}, \quad \ddot{\mathbf{r}}_2 = \frac{Gm_1}{|\mathbf{D}|^3}\mathbf{D}, \tag{1}$$

where $\mathbf{r}_i$ are position vectors of particles with indices $i = 1, 2$ and $\mathbf{D} = \mathbf{r}_1 - \mathbf{r}_2$. Setting $\mu \equiv Gm_1 = -Gm_2$ we obtain from (1)

$$\dot{\mathbf{V}} = \frac{\mu}{|\mathbf{D}|^3}\mathbf{D}, \quad \dot{\mathbf{v}} = 0, \tag{2}$$

where $\mathbf{V} \equiv \frac{1}{2}(\dot{\mathbf{r}}_1 + \dot{\mathbf{r}}_2)$ and $\mathbf{v} \equiv \dot{\mathbf{D}}$. From equation (2) it follows that when $\mathbf{v} = 0$ at some moment of time it remains zero in the course of evolution of the system. Thus, in this case the relative separation distance $\mathbf{D}$ does not change during the evolution and the system ever accelerates as a whole, with the acceleration vector

$$\mathbf{a} \equiv \dot{\mathbf{V}} = \frac{\mu}{|\mathbf{D}|^3}\mathbf{D} \tag{3}$$

being constant. The laws of conservation are nonetheless respected since the kinetic energy and momentum of the system are precisely zero while the potential energy depends only on the relative separation distance[1].

When $\mathbf{v}(t=0) \equiv \mathbf{v}_{in} \neq 0$ the absolute value of the relative distance changes with time. Accordingly, the absolute value of the acceleration changes as well and eventually decays provided that $(\mathbf{D}_{in} \cdot \mathbf{v}_{in}) \neq -|\mathbf{v}_{in}||\mathbf{D}_{in}|^2$, where $\mathbf{D}_{in} \equiv \mathbf{D}(t=0)$. We have

$$\mathbf{D} = \mathbf{v}_{in} t + \mathbf{D}_{in}, \tag{4}$$

[1] Note that it is easy to show that the same motion can be realised in a system containing N particles provided that the total mass of the system $M = \sum_{i=1,n} m_i = 0$ and positions of the particles are chosen in a special way. Say, for a system containing three particles their relative positions must form an equilateral triangle.

[2] Clearly, when $(\mathbf{D}_{in} \cdot \mathbf{v}_{in}) = -|\mathbf{v}_{in}||\mathbf{D}_{in}|$ the particles collide.

and, thus, integrating equation (3) we obtain

$$|\mathbf{V}(t)| = \frac{1}{\sqrt{1-\alpha^2}} \frac{\mu}{D_{in} v_{in}} \sqrt{2(1 - \frac{\epsilon^{1/2} + \alpha\tau}{\Delta})}, \tag{5}$$

where we use the dimensionless time $\tau = \sqrt{\frac{\mu}{D_{in}^3}} t$, $D_{in} = |\mathbf{D}_{in}|$, $v_{in} = |\mathbf{v}_{in}|$, $\epsilon = \frac{\mu}{D_{in} v_{in}^2}$, $\alpha = (\mathbf{v}_{in} \cdot \mathbf{D}_{in})/(v_{in} D_{in})$, and $\Delta = \sqrt{\epsilon + \tau^2 + 2\alpha\epsilon^{1/2}\tau}$.

Note that when the system moves along a straight line with increasing value of $|\mathbf{D}|$, and, accordingly, $\alpha = 1$ equation (5) yields

$$|\mathbf{V}(t)| = \frac{\mu}{D_{in} v_{in}} \frac{\tau}{\epsilon^{1/2} + \tau}. \tag{6}$$

In the limit $\tau \to \infty$ we get from equations (5) and (6)

$$V_\infty \equiv |\mathbf{V}(\tau \to \infty)| = \sqrt{\frac{2}{1+\alpha}} \frac{\mu}{D_{in} v_{in}}. \tag{7}$$

It follows from (7) that when

$$v_{in} < v_{crit} = \sqrt{\frac{2}{1+\alpha}} \frac{\mu}{D_{in} c}, \tag{8}$$

the asymptotic value of velocity of the system, V_∞, formally exceeds the speed of light, c. Clearly, a relativistic approach to the problem is to be used in this situation.

III. RELATIVISTIC TREATMENT

A. Derivation of dynamical equations

In order to keep our study as simple as possible let us consider in the relativistic case only the motion along a straight line with increasing value of $\mathbf{D}$ ($\alpha = 1$). Additionally, in this Section we use the natural units setting the speed of light and the gravitational constant to unity. However, unlike the Newtonian case, here we would like to consider particles having a small mass difference: $m_{1,2} = \mu \pm \Delta\mu$, where it is assumed below that $\mu > 0$ and $|\Delta\mu| \ll \mu$.

It is useful to introduce two local frames and the respective coordinate systems 1) a fixed lab frame with global Lorentzian coordinates (x, t) and 2) a local Lorentzian frame instantaneously comoving with the motion of the point $R(t) = \frac{1}{2}(x_1(t) + x_2(t))$, where $x_1(t)$ and $x_2(t)$ are positions of the particles in the lab frame, with associated Lorentzian coordinates (x^{com}, t^{com}). Is is assumed that at some particular moment of time $t = t_*$ coordinates of the event $(t_*, R(t = t_*))$ in the comoving coordinate system are equal to $(\tau, 0)$, where τ is the proper time associated with the world line $(t, R(t))$. Hereafter, the world line $(t, R(t))$ is referred to as "the reference world line".

When $t^{com} = \tau$ the positions of particles are given by $x_{1,2}^{com}(t^{com})$, their velocities are $v_{1,2}^{com} = \frac{d}{dt^{com}} x_{1,2}^{com}$. Let us also introduce the relative position and velocity in the comoving coordinate system $D = x_1^{com} - x_2^{com}$, $v^{com} = \frac{d}{dt^{com}} D$. Without loss of generality we assume hereafter $D^{com} > 0$. When the relative separation remains sufficiently small along the reference world line we have approximately $x_2^{com} = -x_1^{com}$.

In the global coordinates at the time slice $t = t_*$ the velocity of motion of the system as a whole is given by $V = \frac{1}{2}(\frac{d}{dt}x_1 + \frac{d}{dt}x_2)(t = t_*)$ while the relative position and velocity of the relative motion are $D_{lab} = x_1(t_*) - x_2(t_*)$ and $v = \frac{d}{dt} D$.

Introducing the Lorentz gamma factor $\gamma = \frac{1}{\sqrt{1-V^2}}$ associated with the reference world line we may write in the limit of small separations

$$D(t^{com}) = \gamma D_{lab}, \quad \frac{dt}{dt^{com}} = \gamma, \tag{9}$$

and, accordingly,

$$v^{com} = \gamma \frac{d}{dt}(\gamma D_{lab}) = \gamma^2 v + \gamma \frac{d\gamma}{dt} D_{lab}. \tag{10}$$

Supposing below that, on one hand, the relative distance $D \gg \mu$, and, therefore, a weak-field approximation holds and, on the other hand, it is not too large for the local Lorentzian coordinates to be adequate and, respectively, equations (9-10) to be valid we can use the Newtonian expression for the energy, E_c, and momentum, P_c, of the system in the comoving frame at the time $t^{com} = \tau$

$$E_c = 2\Delta\mu + \frac{\mu^2}{D}, \quad P_c = \mu\dot{D}, \tag{11}$$

where dot stands for differentiation w.r.t. the proper time τ.

In the same limit E_c and P_c represent time and spacial components of a local four vector, and, therefore, they values in the lab frame, E and P, respectively, can be obtained from (11) by the standard Lorentz transformation. We have

$$E = \gamma(2\Delta\mu + \frac{\mu^2}{D} + V\mu\dot{D}), \quad P = \gamma(\mu\dot{D} + V(2\Delta\mu + \frac{\mu^2}{D})), \tag{12}$$

where it is assumed that the velocity of the systems as a whole, V, is a function of the proper time τ. Since energy and momentum in the lab frame are obviously conserved, equations (12) fully describe the dynamics of our system. They should be solved subject to the condition that the system is initially at rest with respect to the lab frame: when $\tau = 0$ we have $V = 0$ and

$$E = E_{in} = 2\Delta\mu + \frac{\mu^2}{D_{in}}, \quad P = P_{in} = \mu v_{in}, \tag{13}$$

where D_{in} and v_{in} are initial separation distance and relative velocity, respectively. It is assumed below that $v_{in} > 0$.

Although our derivation of dynamical equations (12) may look somewhat heuristic it is worth mentioning that when terms next to the leading order in μ are discarded they can be derived from the precise weak-field equations of reference [4] in the limit of small separations and $|\Delta\mu| \ll \mu$.

It is convenient to transform equations (12) to another form using their linear combination $E - VP$ and calculating square of the norm of the energy-momentum vector, $N^2 = E^2 - P^2$. We get

$$E - VP = \gamma^{-1}(2\Delta\mu + \frac{\mu^2}{D}) \tag{14}$$

and

$$N^2 = (2\Delta\mu + \frac{\mu^2}{D})^2 - \mu^2(\dot{D})^2. \tag{15}$$

We also obviously have $N^2 = (2\Delta\mu + \frac{\mu^2}{D_{in}})^2 - \mu^2 v_{in}^2$. Note that contrary to the usual situation the energy-momentum vector can be null, time-like or space-like, depending on initial conditions.

Equations (12) are first order integrals of two second order in time dynamical equations. One of these equations can be obtained from (15) by differentiating this equation over τ with the result

$$\ddot{D} = -\frac{2\Delta\mu}{D^2} - \frac{\mu^2}{D^3}, \tag{16}$$

and the second one by differentiating either of equations (12) and using (16):

$$\gamma^2\dot{V} = \frac{\mu}{D^2} \tag{17}$$

Equations (16) and (17) can be obtained from other independent qualitative arguments. The derivation of the second order dynamical equations, which relate dynamical variables with different values of time coordinates is not, however, convenient in the local Lorentzian coordinates introduced above since these coordinates are defined with respect to some particular event on the reference world line and, therefore, the definition is different for different events along this world line. It is much more convenient to use a coordinate system, where the proper time τ plays the role of coordinate time. For that let us consider another, the so-called local Fermi-Walker coordinate system (τ, y), see e.g. [5], where the proper time τ is the coordinate time and the unit vector in the spacial direction y is always perpendicular to the four velocity along the reference world line. The coordinates of the reference world line in this coordinate system are simply $(\tau, 0)$.

At the time slice $t^{com} = \tau$ the local Lorentz coordinates and the Fermi-Walker coordinates coincide: $x^{com}_{1,2} = y_{1,2}$, but the Fermi-Walker coordinate system is accelerating with respect to the local Lorentz coordinate system with an

acceleration $g(y)$. Clearly, $g(y=0)$ must coincide with modulus of four acceleration of the reference world line with respect to the lab frame. The equations of motion in the Fermi-Walker coordinates are assumed to be determined by the Newton's law (1) with added acceleration term $-g$, which accounts for the fact that this system is not inertial:

$$\ddot{y}_{1,2} = \frac{\mu \mp \Delta\mu}{D_{FW}^2} - g, \tag{18}$$

where $D_{FW} = y_1 - y_2$ and we take into account that the acceleration term depends on the coordinate y, e.g. [5]: $g = a + a^2 y$. For the average distance $Y = (y_1 + y_2)/2$ to be at rest $Y(\tau) = 0$ the acceleration term a must be balanced by the gravity term $\frac{\mu}{D_{FW}^2}$. We get

$$a = \frac{\mu}{D_{FW}^2}. \tag{19}$$

Taking into account that in the lab frame the spacial coordinate of four acceleration is related to a as $a^x = \gamma a$ we have

$$\dot{U}^x = \gamma^3 \dot{V} = \gamma a, \quad \dot{V} = \frac{\mu}{\gamma^2 D_{FW}^2}. \tag{20}$$

It is clear that this coincides with equation (17).

The dynamical equation for the relative distance D_{FW} directly follows from (18) and (19):

$$\ddot{D}_{FW} = -\frac{2\Delta\mu}{D_{FW}^2} - a^2 D_{FW} = -\frac{2\Delta\mu}{D_{FW}^2} - \frac{\mu^2}{D_{FW}^3}. \tag{21}$$

It coincides with (16).

The last term on the right hand side of (21) is due to the non-uniform acceleration force appearing in the Fermi-Walker coordinates. Because it is $\propto \mu^2$, technically, it is a post-Newtonian term. Since we consider the gravitational force in the Newtonian approximation in (21) it is important to check whether or not post-Newtonian corrections to the gravitational force are comparable with the acceleration term in (21). In fact, as is described in standard handbooks, e.g. [6], the post-Newtonian corrections are either proportional to $\Delta\mu$ or $\dot{y}_{1,2}$. The mass difference and velocities are assumed to be small and therefore, the terms in (21) arising from the post-Newtonian corrections appear to be small compared to the terms taken into account.

From (21) it follows that when the mass difference is negative and $D_{FW} = 2|\Delta\mu|$ the particles are at rest with respect to each other. In this case the Fermi-Walker coordinate system locally coincide with the Rindler one and the particles accelerate indefinitely. Thus, unlike the Newtonian case considered in the previous Section the particles accelerating indefinitely being at rest with respect to each other must have the small mass difference. This effect was first noted by Bondi 1957 [1]. It is obviously due to the non-uniform character of the acceleration term.

B. Solution of dynamical equations

Since equation (14) contains only V and D it can be used to express V in terms of D

$$V = \frac{EP}{E_c^2 + P^2}\left(1 \mp \frac{E_c}{EP}\sqrt{E_c^2 - N^2}\right), \tag{22}$$

where E_c is expressed through D in equation (11), E and P are given in equation (13). As we discussed above we assume that at the initial moment of time $t = \tau = 0$ we have $V = 0$. That means that initially we have to choose the sign $(-)$ in (22). However, under certain conditions discussed below the direction of motion of the particles relative to each other and, accordingly, $\dot{D}$, changes sign. At the turning point $\dot{D} = 0$ we have $N^2 = E_c^2$. Since velocity V must grow monotonically according to (17) we must take the sign $(+)$ in (22) after the turning point.

On the other hand equation (15) contains only D and its derivative with respect to the time τ, and, therefore, it can be integrated to obtain the dependence of D on time. Explicitly we have

$$\int_{D_{min}}^{D} \frac{x dx}{\sqrt{R(x)}} = \tau/\mu, \tag{23}$$

where

$$R(x) = (\mu^2 + 2\Delta\mu x)^2 - N^2 x^2. \tag{24}$$

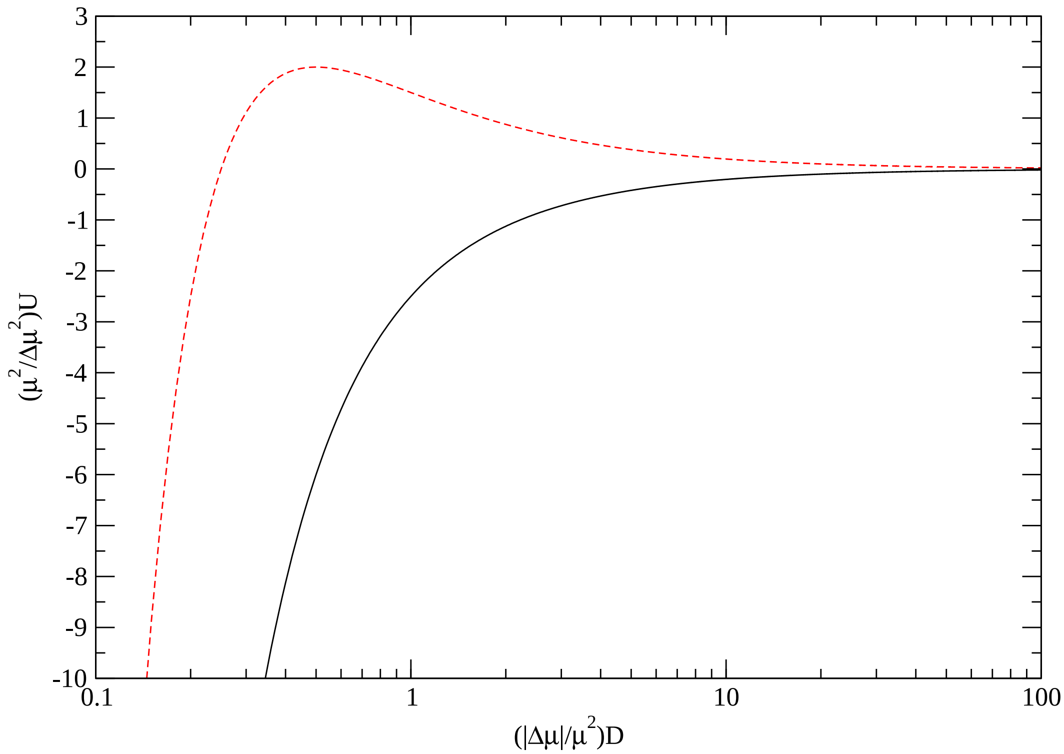

FIG. 1: The dependence of the potential U on the spacial coordinate D. The solid curve corresponds to the case $\Delta\mu > 0$ while the dashed one to the case $\Delta\mu < 0$.

The integral in (23) can be evaluated by a standard substitution to give an explicit relation between τ and D. However, the final expressions are rather cumbersome and we do not show them here. Instead, in general, we analyse qualitatively solutions to (15) based on analogy between this equation and an equation describing a motion of a particle in a potential well. For that we bring (15) to a standard form

$$\frac{\dot{D}^2}{2} + U(D) = \mathcal{E}, \quad U(D) = -\frac{2\Delta\mu}{D} - \frac{\mu^2}{2D^2}, \tag{25}$$

where

$$\mathcal{E} = \frac{4\Delta\mu^2 - N^2}{2\mu^2} = \frac{v_{in}^2}{2} - \frac{2\Delta\mu}{D_{in}} - \frac{\mu^2}{2D_{in}^2}. \tag{26}$$

Introducing natural units $\tilde{U} = \frac{\mu^2}{\Delta\mu^2}U$ and $\tilde{D} = \frac{|\Delta\mu|}{\mu^2}D$ we can express $\tilde{U}$ in terms of $\tilde{D}$ in a very simple form: $\tilde{U} = \mp\frac{2}{\tilde{D}} - \frac{1}{2\tilde{D}^2}$, where the sign $-$ $(+)$ corresponds to $\Delta\mu > 0$ $(\Delta\mu < 0)$. The dependence $\tilde{U}(\tilde{D})$ is shown in Fig. 1.

At first let us consider in detail an important case of zero norm of the energy-momentum vector, $N^2 = 0$, and set, accordingly, $P = E$. A simple analysis of equation (22) shows that in this case there are no turning points, the relative separation D grows with time and the value of $V = 1$ can be achieved in the asymptotic limit $\tau \to \infty$. Therefore, in this case the system may accelerate indefinitely.

When $N^2 = 0$ equation (22) simplifies to

$$V = \frac{E^2 - E_c|E_c|}{E^2 + E_c^2}, \quad \gamma = \frac{(E^2 + E_c|E_c|)}{2EE_c} \tag{27}$$

and from equation (23) we get

$$\tau = \frac{1}{4\Delta\mu^2}(2\Delta\mu(D - D_{min}) - \mu^2\log(\frac{\mu^2 + 2\Delta\mu D}{\mu^2 + 2\Delta\mu D_{min}})). \tag{28}$$

From equation (27) it follows that when $E_c > 0$ the indefinite acceleration is possible only if $E_c \to 0$ when $\tau \to \infty$ and from the expression for E_c (11) it is seen that the mass difference $\Delta\mu$ must be negative for that. We consider below only this case in detail. When $|\Delta\mu| \neq 0$ $E_c \to 0$ provided that $D \to D_{crit} = \mu^2/(2|\Delta\mu|)$. Equation (28) tells that the logarithm on the right hand side diverges when $D \to D_{crit}$. That means that this limit does correspond to the limit $\tau \to \infty$. Let us estimate the dependence of the Lorentz factor γ on time in this case. To do so, we introduce a new variable $\Delta = D_{crit} - D$ and substitute it to (28) assuming that it is small. We get

$$\tau \approx \frac{\mu^2}{4\Delta\mu^2}\log(\frac{\mu^2 - 2|\Delta\mu|D_{in}}{2|\Delta\mu|\Delta}), \tag{29}$$

and, substituting this result into equation (27) we have

$$\gamma \approx \frac{\mu^2}{4|\Delta\mu|D_{in}}\exp\frac{4\Delta\mu^2}{\mu^2}\tau. \tag{30}$$

Equation (30) tells that when $D \approx D_{crit}$ acceleration is exponentially fast.

The degenerate case $\Delta\mu = 0$ must be analysed separately. In this case from (28) we have

$$\tau = \frac{1}{2\mu^2}(D^2 - D_{min}^2), \tag{31}$$

and the distance D increases indefinitely with time. From equation (27) we obtain

$$\gamma \approx \frac{\mu}{2D_{min}}\sqrt{2\tau}. \tag{32}$$

Now let us turn to the general case $N^2 \neq 0$. Setting $\dot{D} = 0$ in (25) we get a general equation for the turning points

$$D_{1,2} = \frac{\Delta\mu}{\mathcal{E}}(-1 \pm \sqrt{1 - \frac{\mu^2\mathcal{E}}{2\Delta\mu^2}}) = \frac{\Delta\mu}{\mathcal{E}}(-1 \pm \frac{\sqrt{N^2}}{2\Delta\mu}). \tag{33}$$

Equation (33) tells that the turning points exist only when $N^2 > 0$. Their number depends on signs of $\mathcal{E}$ and $\Delta\mu$. When $\Delta\mu > 0$ the potential $U(D)$ is negative, see Fig. 1, and therefore, the relative motion is finite for $\mathcal{E} < 0$ with one turning point [3]

$$D_1 = \frac{\Delta\mu}{|\mathcal{E}|}(1 + \frac{N}{2\Delta\mu}). \tag{34}$$

In the opposite case $\mathcal{E} > 0$, and, accordingly, $N < 2\Delta\mu$, the motion is unbound and the relative distance D grows indefinitely with time.

When $\Delta\mu < 0$ the potential $U(D)$ acquires positive values for $D > \frac{\mu^2}{4|\Delta\mu|}$, see Fig. 1. It tends to zero when $D \to \infty$ and has a maximum at $D = D_{crit}$. Note that from the condition $U(D_{crit}) = \mathcal{E} = \frac{2\Delta\mu^2}{\mu^2}$ we get $N^2 = 0$ there. The character of the relative motion depends on whether $\mathcal{E}$ is negative, belongs to the interval $0 < \mathcal{E} < \frac{2\Delta\mu^2}{\mu^2}$ corresponding to $0 < N < 2|\Delta\mu|$, or $\mathcal{E} > \frac{2\Delta\mu^2}{\mu^2}$ and, accordingly, $N^2 < 0$. When the energy $\mathcal{E}$ is negative the motion is bound with one turning point

$$D_1 = \frac{|\Delta\mu|}{|\mathcal{E}|}(-1 + \frac{N}{2\Delta\mu}). \tag{35}$$

In the intermediate region $0 < \mathcal{E} < \frac{2\Delta\mu^2}{\mu^2}$ there are two turning points

$$D_\pm = \frac{|\Delta\mu|}{\mathcal{E}}(1 \pm \frac{N}{2\Delta\mu}). \tag{36}$$

[3] Let us remember that we consider only positive values of D.

When $D_{in} < D_-$ the motion is bound while for $D_{in} > D_+$ D grows indefinitely. Finally, when $N^2 < 0$ the motion is always unbound.

When the motion is bound the velocity $\dot{D}$ changes sign after the turning point and in this case we should use the sign (+) in (22). Taking into account that D is decreasing after the turning point and that $E_c \propto D^{-1}$ we see from (22) that the velocity $V \to 1$. The particles tend to collide. However, our assumption that $D \gg \mu$ breaks down in this case and we cannot describe the motion at scales $D \sim \mu$ within the framework of our formalism. Note that we consider in this study only pairs of particles with strictly zero angular momentum. In the situation when the particles have a small but nonzero angular moment they would miss each other and after certain moment of time the distance D would become negative. In this case the analysis of this paper can be repeated without any major change for negative values of D and one would conclude that for such parameters of motion there is another symmetric turning point at negative values of D. Thus, the relative motion of a pair of particles with small but nonzero orbital momentum would be periodic much similar to the case of ordinary particles with positive masses.

Now let us consider the case of the unbound motion and estimate the maximal value of the Lorentz gamma factor the system can reach. As follows from our previous discussion when $N^2 \neq 0$ the distance D grows indefinitely. That means that the energy in the comoving frame, E_c, must tend asymptotically to $2\Delta\mu$. Note that when $\Delta\mu < 0$ the asymptotic value of E_c is negative. We have from (22) setting $E_c = 2\Delta\mu$ there

$$V = \frac{1}{4\Delta\mu^2 + E^2 - N^2}(E\sqrt{E^2 - N^2} - 2\Delta\mu\sqrt{4\Delta\mu^2 - N^2}). \tag{37}$$

Equation (37) tells that when $\Delta\mu > 0$ the last term in the brackets is negative and the asymptotic value of velocity is smaller than 1. Large values of V can be achieved in the opposite case $\Delta\mu < 0$ assuming $|N^2| \ll \Delta\mu^2$. In this case we expand expressions in (37) in the Taylor series in $|N^2|/\Delta\mu^2$ to obtain

$$V = 1 - \frac{N^4}{32\Delta\mu^2 E^2}, \tag{38}$$

and, accordingly

$$\gamma \approx \frac{1}{\sqrt{2(1-V)}} = \frac{4|\Delta\mu E|}{|N^2|}. \tag{39}$$

Equation (39) tells that for fixed values of E and $\Delta\mu < 0$ the gamma factor can be made arbitrary large by choosing arbitrary small values of $|N^2|$. This conclusion is in agreement with our previous finding that the system accelerates indefinitely when $N^2 = 0$.

IV. METHODOLOGICAL COMMENTS

Here I would like to make comments on several methodological issues related to the problem.

1) At first glance the fact that the 'average' position of the pair $(x_1 + x_2)/2$ always grows with time may seem to be in contradiction with the law of conservation of the centre of mass of the system. This contradiction is resolved by observation that for the system containing particles of opposite masses position of the centre of mass, R, is determined by a *difference* of positions of particular particles. Say, in the Newtonian approximation we have $R = m_1x_1 + m_2x_2 = (\mu + \Delta\mu)x_1 - (\mu - \Delta\mu)x_2$. In the relativistic case the situation is analogous for systems with $N^2 > 0$. In the opposite case the notion of centre of masses is ill defined. Indeed, introducing the velocity of a coordinate system, where the centre of mass is at rest in a standard way as $V_{cm} = P/E$ [6] we see that when $N^2 = 0$ $V_{cm} = 1$ and when $N^2 < 0$ V_{cm} formally exceeds the speed of light. It is obvious that the notion of the centre of mass is redundant in both cases.

2) In Introduction of their paper the authors of [3] claim that the conception of PMT put forward by the author of this note is related to the problem of indefinite acceleration of two gravitationally interacting particles. This statement needs, in my opinion, a clarification. First, let me note that as it is discussed above even in the case when only a finite acceleration of the particles is attained PMT is still possible in a situation where production of such pairs is provided by some physical mechanism. Second, the conception of PMT, in general, does not rely on gravitation interactions at all. In particular, in paper [2] I consider a model where there is a continuous flow of positive energy from some spacial regions having negative energy to other regions with positive energy provided by hydrodynamical effects. In this model the space-time is assumed to be flat and gravitational interactions are absent. Moreover, in order to construct a PMT it is not necessary to invoke objects having negative rest masses. It is enough to consider a medium with positive comoving energy density violating the weak energy condition [2]. Additionally, there are ways of constructing PMT,

where gravitational interaction plays a totally different role, say, transferring the energy from a non-stationary system having negative mass to gravitational waves, as for example in the model of a rotating relativistic string connected by two negative mass monopoles [2], [7]. The effects related to the dynamics of free negative mass particles are clearly irrelevant to such systems.

3) The authors of [3] claim that it is impossible to obtain, in principal, an indefinite acceleration of the system containing two particles with opposite signs of masses. One may think that this clearly contradicts to the Bondi's result [1] and the results of this paper. The conundrum is resolved by observation that the authors of [3] consider only *relative* motions while Bondi's analysis as well the analysis in this note also deal with the motion of the pair of particles as a whole with respect to an inertial frame.

V. CONCLUSIONS

In this note we show by elementary means that in the weak limit approximation a pair of particles having opposite values of masses can be accelerated indefinitely provided that the energy-momentum vector characterising the system is null. The system can also be accelerated to arbitrary large Lorentz factors when the mass difference $\Delta\mu < 0$ and the norm of the energy-momentum vector is sufficiently small.

Assuming that there is a process of production of such pairs and that the positive mass particles are intercepted with a target while the negative mass particles fly away it is possible to transfer to the target any desired amount of energy. In a more natural situation one can also consider a theory where the positive and negative mass particles interact differently with a normal matter. A general situation of this kind where there is a persistent transfer of energy from a subsystem having negative or almost zero energy (like this pair) to a subsystem with positive energy was dubbed by us 'Perpetuum Motion of Third Kind' (PMT) [2]. Note, however, that it is just a classical analog of the well known instability of a quantum system with a number of negative energy states unbound from below.

The question of whether the existence of PMT or ever accelerating pairs of particles is a paradox depends, in our opinion, on definition of what paradox is. On one hand, for example, Bonnor 1989 states 'I regard the runaway (or self-accelerating) motion ... so preposterous that I prefer to rule it out by supposing that inertial mass is all positive or all negative' [8]. Clearly, existence of PMT can also be classified as a kind of runaway. On the other hand, no laws of physics are broken in such systems. We believe that the existence of runaways of these kinds is dangerous for theories where they present. To exemplify, an indefinite concentration of energy of different signs in spatially separated regions could lead to a highly inhomogeneous space-time hardly compatible with presence of any life. Therefore, such theories should be ruled out though some additional study of them in General Relativity may be of some interest.

Since in our approximation only linear metric perturbations and one next-to-the-leading order term determined by the acceleration of the pair as a whole are taken into account, it is interesting to estimate what kind of corrections can be obtained by considering other higher order terms quadratic in metric perturbations? For a non-relativistic motion with $V \ll c$ for this purpose one can use the well known Einstein-Infeld-Hoffmann equations of motion (e.g. [6]). In this way it is convenient to consider particles with a large mass difference as well as systems with non-zero angular momentum. There are, however, many corrections, which are absent in such a treatment, notably the emission of gravitational waves. Therefore, a self-consistent relativistic treatment of the problem in the next to the weak field approximation must be based on the second-order formalism of Havas and Goldberg 1962. Such an approach is left for a possible future work.

Although in this paper we consider only particles with no internal structure our analysis may also be valid for a pair of extended objects with total energies of opposite signs provided that they have a sufficiently large separation distance and that their relative velocities are sufficiently small. For example, Deser and Pirani [9] considered the behaviour of systems with all possible inertial/gravitational mass signs and noted that a pair of geons having opposite signs of their total energies would behave as a pair of point particles in the corresponding limit.

It is also interesting to point out that the notion of 'Perpetuum Motion of Third Kind' was introduced in the context of thermodynamical systems having negative temperatures, where one can withdraw heat from a negative temperature reservoir and convert it completely to work, see e. g. [10], p. 176. Since thermodynamical systems with negatives masses of their components should have negative temperatures (e.g. [11]) there is a link between thermodynamical properties of such systems and the ones discussed in this paper. In particular, a runaway process occurring in a thermodynamical system having two subsystems containing particles with opposite signs of masses has been discussed in e.g. [12]. It has been mentioned that this process is analogous to the self-acceleration of a pair of particles with opposites signs of their masses.

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